

A NEW CODE FOR NONRADIAL STELLAR PULSATIONS AND ITS APPLICATION TO LOW-MASS, HELIUM WHITE DWARFS

Letter to the Editor

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Abstract. We present a finite difference code intended for computing linear, adiabatic, nonradial pulsations of spherical stars. This code is based on a slight modification of the general Newton-Raphson technique in order to handle the relaxation of the eigenvalue (square of the eigenfrequency) of the modes and their corresponding eigenfunctions. This code has been tested computing the pulsation spectra of polytropic spheres finding a good agreement with previous work.

Then, we have coupled this code to our evolutionary code and applied it to the computation of the pulsation spectrum of a low mass, pure-helium white dwarf of $0.3 M_{\odot}$ for a wide range of effective temperatures. In making this calculation we have taken an evolutionary time step short enough such that eigenmodes corresponding to a given model are used as initial approximation to those of the next one. Specifically, we have computed periods, period spacing, eigenfunctions, weight functions, kinetic energies and variational periods for a wide range of modes. To our notice this is the first effort in studying the pulsation properties of helium white dwarfs. The solution we have found working with these realistic white dwarf models are in good accord with the predictions of the asymptotic theory of Tassoul (1980) for high order modes. This indicates that the code presented here is able to work adequately also with realistic stellar models.

1. Introduction

At present, nonradial stellar pulsations constitute a valuable tool in extracting information about the properties and internal structure of objects that undergo such kind of oscillations. As observational techniques are refined, a growing variety of stellar objects is discovered that undergo nonradial oscillations. The most extensively studied case is our Sun, which has been shown to vibrate in a large amount of modes, investigation of which has been so powerful in revealing clues of its internal structure that it is known as the discipline of *Helioseismology*. Also, it has been established that different kinds of stellar objects which are located in a variety of places in the HR diagram, covering several evolutionary stages, undergo nonradial pulsations, e.g.: Ap, WR, δ Scuti, β Cephei, and variable white dwarfs (hereafter

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WDs): DAV, DBV, DOV and PNNV. The study of oscillations in such variable stars, and the subsequent comparison and fitting with theoretical pulsation patterns is now known as *Asteroseismology*. For a general introduction to this topic see, e.g., Cox (1980); Unno et al. (1989); Brown and Gilliland (1994); and Gaustchy and Saio (1995; 1996).

From the observational point of view (with the obvious exception of our Sun), WDs represent the best established kind of nonradial pulsators. WDs are the final fate for low and intermediate mass stars. During its cooling, they go across very narrow instability strips inside which suffer nonradial pulsations in the branch of g-(gravity) modes with periods between 100 and 2000 sec and amplitudes up to 0.3 magnitudes.

As WDs cools down, the outermost layers main constituent (hydrogen in DAVs, helium in DBVs, and probably carbon or oxygen in DOVs and PNNVs; see e.g. Brown and Gilliland (1994) and Gaustchy and Saio (1995; 1996) gets partial ionization conditions which forces the development of an outer convection zone (OCZ). In the bottom of such zone the instabilities are originated (κ mechanism of overstability). This gives rise to the blue edge of the instability strip (see, e.g., Dziembowski and Koester, 1981 and Winget et al., 1982; see also Tassoul, Fontaine and Winget, 1990, hereafter TFW)*. The physical reason for the appearance of a red edge is still under debate.

At present, there is a general consensus that variable WDs are very interesting objects for asteroseismological studies. Their very simple internal structure allows us to predict theoretically the modes with a very high degree of detail and sophistication. Also, they have a very rich spectrum of periods which may give us information about stellar masses (by means of measuring the mean period spacing), core composition (by measuring the rate of change of the period \dot{P}), mass of the surface helium and hydrogen layers (if present) (by means of the departures from uniform period spacing and mode trapping), velocity of rotation and strength of the magnetic field (studying the fine structure of a multiplet), etc. (see, e.g., Bradley, 1996). So, it is not surprising that in the last years DAVs and DBVs as well as DOVs have been the preferred target for the network called 'Whole Earth Telescope' (WET). WET observations have been of an unprecedented quality which in some cases (see, e.g., Winget et al., 1991; Kawaler, 1993; Bradley and Winget, 1994) allowed to apply the powerful tools above mentioned.

In view of the growing importance of this line of research, in our Observatory we have begun the study of nonradial oscillations. To this end, we have developed a code to compute eigenfunctions and eigenfrequencies for spherical objects in the linear, adiabatic approximation. The main purpose of the present paper is to describe the numerical techniques employed in our program to search for and to compute the modes and its application to the case of a $0.3 M_{\odot}$ pure helium WD.

* Recently Goldreich and Wu (1999) have proposed the 'convective driving mechanism' as responsible for overstability of g-modes in DA WDs.

This code is based on a modification of the general Newton-Raphson technique presented in Kippenhahn, Weigert and Hofmeister (1967) to solve the set of difference equations which represent the differential equations of linear, adiabatic, nonradial pulsations of spherical stars.

In order to test the code we have applied it to the case of polytropic spheres whose eigenfrequencies have been computed with very high accuracy (Christensen-Dalsgaard and Mullan, 1994). Also, we have coupled our pulsation code to an stellar evolution code. We have done it in order to be able to follow the changes in the pulsational spectrum of a given model during its evolution with a number of steps as large as it would be needed by specific studies. In making such computation we selected an evolutionary time step short enough such that eigenvalues and eigenfunctions corresponding to a given model are used as a initial approximation to those of the next one. This is very useful for a relaxation code like this and is intended to work as it is the case of the Heneyey technique in stellar evolution.

The rest of the paper is organized as follows: Section 2 is devoted to present the differential equations of linear, adiabatic, nonradial pulsations we have to solve. In Section 3 a detailed description of the general Newton-Raphson technique we employ for computing the eigenmodes is presented. Section 4, presents the results for the case of polytropic spheres, and in Section 5 we describe the work we have done with a realistic helium WD model. Finally, in Section 6 summarize our results.

2. The Differential Equations of Nonradial Pulsations

The differential equations that govern linear, nonradial pulsations of spherical stars in the adiabatic approximation are (Unno et al., 1989):

$$x \frac{dy_1}{dx} = (V_g - 3) y_1 + \left[\frac{\ell(\ell + 1)}{C_1 \omega^2} - V_g \right] y_2 + V_g y_3, \quad (1)$$

$$x \frac{dy_2}{dx} = (C_1 \omega^2 - A^*) y_1 + (A^* - U + 1) y_2 - A^* y_3, \quad (2)$$

$$x \frac{dy_3}{dx} = (1 - U) y_3 + y_4, \quad (3)$$

$$x \frac{dy_4}{dx} = U A^* y_1 + U V_g y_2 + \left[\ell(\ell + 1) - U V_g \right] y_3 - U y_4. \quad (4)$$

x is the independent variable defined as $x = r/R$ where r is the radial coordinate and R is the stellar radius. The dependent variables are defined as

$$y_1 = \frac{\xi_r}{r}, \quad (5)$$

$$y_2 = \frac{1}{g r} \left(\frac{p'}{\rho} + \Phi' \right), \quad (6)$$

$$y_3 = \frac{1}{g r} \Phi', \quad (7)$$

and

$$y_4 = \frac{1}{g} \frac{d\Phi'}{dr}, \quad (8)$$

$$\omega^2 = \frac{\sigma^2 R^3}{G M_\star}. \quad (9)$$

ξ_r represents the radial displacement of the fluid and p' and Φ' are the Eulerian variations of the equilibrium values of the pressure p and the gravitational potential Φ respectively. g is the local acceleration of gravity and M_\star is the stellar mass. ω^2 is the dimensionless square of the angular frequency of oscillation, σ^2 .

Among the coefficients of Equations (1–4) there appears the harmonic degree ℓ , corresponding to the spherical harmonic $Y_{\ell,m}(\theta, \phi)$ which describes the angular dependence of the oscillation pattern. The adimensional quantities V_g , U , C_1 , and A^* , inherent to the non-perturbed model, are defined as:

$$V_g = \frac{V}{\Gamma_1} = - \frac{1}{\Gamma_1} \frac{d \ln p}{d \ln r} = \frac{g r}{c^2} = \frac{g r \rho}{\Gamma_1 p}, \quad (10)$$

$$U = \frac{d \ln M_r}{d \ln r} = \frac{4\pi \rho r^3}{M_r}, \quad (11)$$

$$C_1 = \left(\frac{r}{R} \right)^3 \frac{M_\star}{M_r}, \quad (12)$$

$$A^* = \frac{r}{g} N^2 = r \left(\frac{1}{\Gamma_1} \frac{d \ln p}{dr} - \frac{d \ln \rho}{dr} \right). \quad (13)$$

Here Γ_1 is the first adiabatic index, c^2 is the square of the local velocity of sound, M_r is the mass contained in a (non-perturbed) sphere of radius r , and N^2 is the Brunt-Väisälä frequency.

Equations (1–4) together with the adequate boundary conditions conform the eigenvalue problem to be solved.

3. The Numerical Code

In order to write the pulsation equations in a form adequate for numerical calculations we divide our non-perturbed model in a number of spherical, concentric shells. This is equivalent to divide the dominion of the independent variable in N mesh points ($N - 1$ shells) non-necessarily equidistantly spaced x_j , $j = 1, \dots, N$. In our treatment we define $x_1 = 1$ as the surface and $x_N = 0$ as the central point of the model.

Now, we replace Equations (1–4) by difference equations. The system of equations may be rewritten as

$$\frac{dy_i}{dx} = f_i(y_1, y_2, y_3, y_4, \lambda), \quad i = 1, 2, 3, 4, \quad (14)$$

where $\lambda = \omega^2$. In finite differences, Equation (14) may be written as*

$$\frac{[y_i]_{j+1} - [y_i]_j}{x_{j+1} - x_j} = f_i([y_1, y_2, y_3, y_4]_{j+\frac{1}{2}}; \lambda); \quad (16)$$

$$i = 1, 2, 3, 4; j = 1, \dots, N - 1$$

where, for any quantity ψ ,

$$[\psi]_{j+\frac{1}{2}} = \frac{[\psi]_j + [\psi]_{j+1}}{2}, \quad (17)$$

being $[\psi]_j$ the value of the quantity ψ at the meshpoint j . The outer boundary conditions are given by** (Unno et al., 1989)

$$\begin{aligned} [y_1]_1 - [y_2]_1 + [y_3]_1 &= 0, \\ (\ell + 1) [y_3]_1 + [y_4]_1 &= 0, \\ [y_1]_1 &= 1. \end{aligned} \quad (18)$$

The last equation is the normalization condition, usually employed in previous works. The central boundary conditions are given by (Unno et al., 1989)

$$\begin{aligned} [y_1]_N \frac{[C_1]_N \lambda}{\ell} - [y_2]_N &= 0, \\ \ell [y_3]_N - [y_4]_N &= 0. \end{aligned} \quad (19)$$

* We have also employed finite differences of the form

$$\frac{[y_i]_{j+1} - [y_i]_j}{x_{j+1} - x_j} = \frac{1}{2} (f_i([y_1, y_2, y_3, y_4]_j; \lambda) + f_i([y_1, y_2, y_3, y_4]_{j+1}; \lambda)); \quad (15)$$

$$i = 1, 2, 3, 4; j = 1, \dots, N - 1.$$

Applying such form for the difference equations we have found results negligibly different from those presented in this paper.

** The first two of these equations are the so called 'zero boundary condition', which we shall employ in the treatment of polytropes. In the case of realistic WD models we have replaced the first one by Equation (30) (see below).

In order to solve this system of difference equations we shall use the general Newton-Raphson technique closely following the formulation presented in Kippenhahn et al. (1967) for the case of stellar evolution. We shall assume we are able to get an approximate solution for the system and we want to improve it iteratively. In the case that this initial solution is not far from the actual solution of the system, we may expand the equations up to first order in the corrections for the values of the eigenfunctions at each meshpoint and also for the eigenvalue (square of the eigenfrequency). In this standard way we get a linear system of equations in the corrections that must be solved.

In condensed way, the algebraic system of equations for the first order corrections may be expressed as

$$\frac{\partial B_k}{\partial [y_1]_1} \delta[y_1]_1 + \cdots + \frac{\partial B_k}{\partial [y_4]_1} \delta[y_4]_1 + \frac{\partial B_k}{\partial \lambda} \delta\lambda = -B_k; \quad k = 1, 2, 3, \quad (20)$$

$$\begin{aligned} & \frac{\partial G_i^j}{\partial [y_1]_j} \delta[y_1]_j + \cdots + \frac{\partial G_i^j}{\partial [y_4]_j} \delta[y_4]_j + \frac{\partial G_i^j}{\partial [y_1]_{j+1}} \delta[y_1]_{j+1} \\ & + \cdots + \frac{\partial G_i^j}{\partial [y_4]_{j+1}} \delta[y_4]_{j+1} + \frac{\partial G_i^j}{\partial \lambda} \delta\lambda = -G_i^j; \quad i = 1, 2, 3, 4; \\ & j = 1, 2, \dots, N-1, \end{aligned} \quad (21)$$

$$\frac{\partial C_m}{\partial [y_1]_N} \delta[y_1]_N + \cdots + \frac{\partial C_m}{\partial [y_4]_N} \delta[y_4]_N + \frac{\partial C_m}{\partial \lambda} \delta\lambda = -C_m; \quad m = 1, 2, \quad (22)$$

where $\delta[y_i]_j$ are small corrections to the eigenfunction y_i at the j - mesh point, and $\delta\lambda$ stand for the correction to the eigenvalue λ . B_k , G_i^j , and C_m are the values of the difference equations when applied the solution to be iteratively improved. Obviously, all of them must be zero when evaluated with their exact solution.

Now, we have to invert a big matrix, that has non-zero elements only in blocks located on the diagonal and in the last column (because of the derivative with respect to the eigenvalue). Notice that this is an important difference compared with the case of stellar evolution. Thus, in handling the big matrix we need a specific algorithm to solve for the corrections.

Evaluating Equation (20) for $k = 1, 2, 3$ and Equation (21) for $i = 1, 2, 3, 4$ and $j = 1$, we may write the first block of such matrix as

$$\begin{bmatrix} \frac{\partial B_1}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial B_1}{\partial [y_4]_1} & 0 & 0 & 0 \\ \frac{\partial B_2}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial B_2}{\partial [y_4]_1} & 0 & 0 & 0 \\ \frac{\partial B_3}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial B_3}{\partial [y_4]_1} & 0 & 0 & 0 \\ \frac{\partial G^1}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial G^1}{\partial [y_4]_1} & \frac{\partial G^1}{\partial [y_1]_2} & \frac{\partial G^1}{\partial [y_2]_2} & \frac{\partial G^1}{\partial [y_3]_2} \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial G^4}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial G^4}{\partial [y_4]_1} & \frac{\partial G^4}{\partial [y_1]_2} & \frac{\partial G^4}{\partial [y_2]_2} & \frac{\partial G^4}{\partial [y_3]_2} \end{bmatrix} \cdot \begin{bmatrix} \delta[y_1]_1 \\ \delta[y_2]_1 \\ \delta[y_3]_1 \\ \delta[y_4]_1 \\ \delta[y_1]_2 \\ \delta[y_2]_2 \\ \delta[y_3]_2 \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} 0 & -\frac{\partial B_1}{\partial \lambda} & -B_1 \\ 0 & -\frac{\partial B_2}{\partial \lambda} & -B_2 \\ 0 & -\frac{\partial B_3}{\partial \lambda} & -B_3 \\ -\frac{\partial G^1}{\partial [y_4]_2} & -\frac{\partial G^1}{\partial \lambda} & -G^1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ -\frac{\partial G^4}{\partial [y_4]_2} & -\frac{\partial G^4}{\partial \lambda} & -G^4 \end{bmatrix} \cdot \begin{bmatrix} \delta[y_4]_2 \\ \delta\lambda \\ 1 \end{bmatrix} .$$

Now, we define vectors $U_s, V_s, W_s, (s = 1, \dots, 4N - 5)$ such that we may write

$$\begin{bmatrix} \delta[y_1]_1 \\ \delta[y_2]_1 \\ \delta[y_3]_1 \\ \delta[y_4]_1 \\ \delta[y_1]_2 \\ \delta[y_2]_2 \\ \delta[y_3]_2 \end{bmatrix} = \begin{bmatrix} U_1 & V_1 & W_1 \\ U_2 & V_2 & W_2 \\ \vdots & \vdots & \vdots \\ U_7 & V_7 & W_7 \end{bmatrix} \cdot \begin{bmatrix} \delta[y_4]_2 \\ \delta\lambda \\ 1 \end{bmatrix} \tag{24}$$

for this block. The coefficients of this vectors may be calculated easily solving

$$\begin{aligned}
 & \begin{bmatrix} \frac{\partial B_1}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial B_1}{\partial [y_4]_1} & 0 & 0 & 0 \\ \frac{\partial B_2}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial B_2}{\partial [y_4]_1} & 0 & 0 & 0 \\ \frac{\partial B_3}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial B_3}{\partial [y_4]_1} & 0 & 0 & 0 \\ \frac{\partial G_1^1}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial G_1^1}{\partial [y_4]_1} & \frac{\partial G_1^1}{\partial [y_1]_2} & \frac{\partial G_1^1}{\partial [y_2]_2} & \frac{\partial G_1^1}{\partial [y_3]_2} \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial G_4^1}{\partial [y_1]_1} & \cdots & \cdots & \frac{\partial G_4^1}{\partial [y_4]_1} & \frac{\partial G_4^1}{\partial [y_1]_2} & \frac{\partial G_4^1}{\partial [y_2]_2} & \frac{\partial G_4^1}{\partial [y_3]_2} \end{bmatrix} \cdot \begin{bmatrix} U_1 & V_1 & W_1 \\ U_2 & V_2 & W_2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ U_7 & V_7 & W_7 \end{bmatrix} \\
 & = \begin{bmatrix} 0 & -\frac{\partial B_1}{\partial \lambda} & -B_1 \\ 0 & -\frac{\partial B_2}{\partial \lambda} & -B_2 \\ 0 & -\frac{\partial B_3}{\partial \lambda} & -B_3 \\ -\frac{\partial G_1^1}{\partial [y_4]_2} & -\frac{\partial G_1^1}{\partial \lambda} & -G_1^1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ -\frac{\partial G_4^1}{\partial [y_4]_2} & -\frac{\partial G_4^1}{\partial \lambda} & -G_4^1 \end{bmatrix}.
 \end{aligned} \tag{25}$$

From Equation (21), and after short manipulation, we may write an arbitrary block of the big matrix (except the central) in the form

$$\begin{bmatrix} \delta [y_4]_j \\ \delta [y_1]_{j+1} \\ \delta [y_2]_{j+1} \\ \delta [y_3]_{j+1} \end{bmatrix} = \begin{bmatrix} U_{4j} & V_{4j} & W_{4j} \\ U_{4j+1} & V_{4j+1} & W_{4j+1} \\ U_{4j+2} & V_{4j+2} & W_{4j+2} \\ U_{4j+3} & V_{4j+3} & W_{4j+3} \end{bmatrix} \cdot \begin{bmatrix} \delta [y_4]_{j+1} \\ \delta \lambda \\ 1 \end{bmatrix}, \tag{26}$$

where $j = 2, \dots, N - 2$. Now, the coefficients of vectors U_s, V_s, W_s may be evaluated solving the system

$$\begin{aligned}
 & \begin{bmatrix} \alpha_1^j & \frac{\partial G_1^j}{\partial [y_1]_{j+1}} & \frac{\partial G_1^j}{\partial [y_2]_{j+1}} & \frac{\partial G_1^j}{\partial [y_3]_{j+1}} \\ \alpha_2^j & \frac{\partial G_2^j}{\partial [y_1]_{j+1}} & \frac{\partial G_2^j}{\partial [y_2]_{j+1}} & \frac{\partial G_2^j}{\partial [y_3]_{j+1}} \\ \alpha_3^j & \frac{\partial G_3^j}{\partial [y_1]_{j+1}} & \frac{\partial G_3^j}{\partial [y_2]_{j+1}} & \frac{\partial G_3^j}{\partial [y_3]_{j+1}} \\ \alpha_4^j & \frac{\partial G_4^j}{\partial [y_1]_{j+1}} & \frac{\partial G_4^j}{\partial [y_2]_{j+1}} & \frac{\partial G_4^j}{\partial [y_3]_{j+1}} \end{bmatrix} \cdot \begin{bmatrix} U_{4j} & V_{4j} & W_{4j} \\ U_{4j+1} & V_{4j+1} & W_{4j+1} \\ U_{4j+2} & V_{4j+2} & W_{4j+2} \\ U_{4j+3} & V_{4j+3} & W_{4j+3} \end{bmatrix} \\
 & = \begin{bmatrix} -\frac{\partial G_1^j}{\partial [y_4]_{j+1}} & -\beta_1^j & -\gamma_1^j \\ -\frac{\partial G_2^j}{\partial [y_4]_{j+1}} & -\beta_2^j & -\gamma_2^j \\ -\frac{\partial G_3^j}{\partial [y_4]_{j+1}} & -\beta_3^j & -\gamma_3^j \\ -\frac{\partial G_4^j}{\partial [y_4]_{j+1}} & -\beta_4^j & -\gamma_4^j \end{bmatrix},
 \end{aligned} \tag{27}$$

where quantities $\alpha_i^j, \beta_i^j, \gamma_i^j$ are defined as

$$\begin{aligned} \alpha_i^j &= \frac{\partial G_i^j}{\partial [y_4]_j} + U_{4j-3} \frac{\partial G_i^j}{\partial [y_1]_j} + U_{4j-2} \frac{\partial G_i^j}{\partial [y_2]_j} + U_{4j-1} \frac{\partial G_i^j}{\partial [y_3]_j}, \\ \beta_i^j &= \frac{\partial G_i^j}{\partial \lambda} + V_{4j-3} \frac{\partial G_i^j}{\partial [y_1]_j} + V_{4j-2} \frac{\partial G_i^j}{\partial [y_2]_j} + V_{4j-1} \frac{\partial G_i^j}{\partial [y_3]_j}, \\ \gamma_i^j &= G_i^j + W_{4j-3} \frac{\partial G_i^j}{\partial [y_1]_j} + W_{4j-2} \frac{\partial G_i^j}{\partial [y_2]_j} + W_{4j-1} \frac{\partial G_i^j}{\partial [y_3]_j}. \end{aligned} \tag{28}$$

with $i = 1, 2, 3, 4$.

Finally, in the central point of the model, we have $j = N - 1$ in Equation (21) and $m = 1, 2$ in (22). It is easy to show that the last block of the big matrix may be written as

$$\begin{aligned} & \begin{bmatrix} \alpha_1^{N-1} & \frac{\partial G_1^{N-1}}{\partial [y_1]_N} & \frac{\partial G_1^{N-1}}{\partial [y_2]_N} & \frac{\partial G_1^{N-1}}{\partial [y_3]_N} & \frac{\partial G_1^{N-1}}{\partial [y_4]_N} & \beta_1^{N-1} \\ \alpha_2^{N-1} & \frac{\partial G_2^{N-1}}{\partial [y_1]_N} & \frac{\partial G_2^{N-1}}{\partial [y_2]_N} & \frac{\partial G_2^{N-1}}{\partial [y_3]_N} & \frac{\partial G_2^{N-1}}{\partial [y_4]_N} & \beta_2^{N-1} \\ \alpha_3^{N-1} & \frac{\partial G_3^{N-1}}{\partial [y_1]_N} & \frac{\partial G_3^{N-1}}{\partial [y_2]_N} & \frac{\partial G_3^{N-1}}{\partial [y_3]_N} & \frac{\partial G_3^{N-1}}{\partial [y_4]_N} & \beta_3^{N-1} \\ \alpha_4^{N-1} & \frac{\partial G_4^{N-1}}{\partial [y_1]_N} & \frac{\partial G_4^{N-1}}{\partial [y_2]_N} & \frac{\partial G_4^{N-1}}{\partial [y_3]_N} & \frac{\partial G_4^{N-1}}{\partial [y_4]_N} & \beta_4^{N-1} \\ 0 & \frac{\partial C_1}{\partial [y_1]_N} & \frac{\partial C_1}{\partial [y_2]_N} & \frac{\partial C_1}{\partial [y_3]_N} & \frac{\partial C_1}{\partial [y_4]_N} & \frac{\partial C_1}{\partial \lambda} \\ 0 & \frac{\partial C_2}{\partial [y_1]_N} & \frac{\partial C_2}{\partial [y_2]_N} & \frac{\partial C_2}{\partial [y_3]_N} & \frac{\partial C_2}{\partial [y_4]_N} & \frac{\partial C_2}{\partial \lambda} \end{bmatrix} \cdot \begin{bmatrix} \delta [y_4]_{N-1} \\ \delta [y_1]_N \\ \delta [y_2]_N \\ \delta [y_3]_N \\ \delta [y_4]_N \\ \delta \lambda \end{bmatrix} \\ & = \begin{bmatrix} -\gamma_1^{N-1} \\ -\gamma_2^{N-1} \\ -\gamma_3^{N-1} \\ -\gamma_4^{N-1} \\ -C_1 \\ -C_2 \end{bmatrix}. \end{aligned} \tag{29}$$

In this case the quantities $\alpha_i^{N-1}, \beta_i^{N-1}, \gamma_i^{N-1}$ are evaluated from Equations (28) at $j = N - 1$.

We note that Equation (29) can be solved in order to obtain the corrections for eigenfunctions y_1, y_2, y_3 and y_4 in the central point of the object, and for the eigenvalue λ . Also, it must be noted that the correction in eigenfunction y_4 pertaining to the immediate outer mesh point of grid is obtained. This correction, namely $\delta [y_4]_{N-1}$, serve as ‘coupling’ between the points N and $N - 1$. In fact, we may use Equation (26) with $j = N - 2$ in order to obtain the rest of corrections belonging to the eigenfunctions at the mesh point $N - 1$. Next, the employment of this procedure for successive downwards values of j (in Equation 26) using $\delta [y_4]_{j+1}$ as coupling between the quantities belonging to consecutive meshpoints, as well as Equation (24), leads to find the corrections for the eigenvalue and eigenfunctions for the complete model. These corrections are applied to the initial solution and the algorithm is employed iteratively up to the stage at which all the (absolute value)

of the corrections are smaller than some previously adopted value. At this point, the complete set of difference equations that represent Equations (1–4) has been solved.

In order to search for the first approximation to the eigenfunctions and the eigenvalue of a mode, we have applied the method of the discriminant presented in Unno et al. (1989). Specifically, the expression adopted reads $D(\omega) = (\ell + 1) [y_3]_1 + [y_4]_1$. Notice that $D(\omega) = 0$ corresponds exactly to the second outer boundary condition given in Equations (18). We refer the reader to that book for more details.

Obviously, for applying this technique we need to define a grid of meshpoints. In getting an appropriate distribution of meshpoints we have employed a simple recipe. As a first approximation to the solution of the oscillation equations we have computed the eigenfunctions and eigenvalue taking the points at which the equilibrium model is defined. After convergence, we have redistributed the grid asking for the same criterium usually employed in stellar evolution (Kippenhahn et al., 1967): that the relative variation of each eigenfunction inside a zone must be below some prescribed limit (if the eigenfunctions are below some other very small, prescribed value we have asked for absolute differences in place of relative ones). If necessary, our program add or remove meshpoints, using spline interpolation over quantities of non-perturbed model, and lineal interpolation in eigenfunctions, since these will be improved after in the general Newton-Raphson stage.

4. Application to Polytopic Spheres

As a first test for our code, we have applied it to the well-known case of nonradial pulsations of polytropic spheres. Polytopes are particularly adequate for the purpose of testing, because in this case we separate the inaccuracies of the non-perturbed model from those inherent to the nonradial pulsation code. In particular we have computed the p- (pressure), f- (fundamental) and g- (gravity) modes of polytopes with indices $n = 1.5, 2, 2.5, 3, 3.5$ and 4 for the case of $\ell = 2, 3$ and 4.

Because in this case V_g and A^* are divergent at surface (see Equations 10 and 13), we have integrated the structure by means of an accurate Runge-Kutta technique (Press et al., 1992) and most of the meshpoints were located near surface. Here we assumed $\Gamma_1 = \frac{5}{3}$.

To compare our eigenvalues with those available in the literature we shall employ the work of Christensen-Dalsgaard and Mullan (1994) which presents accurate eigenfrequency tabulations for a variety of polytropic spheres. For such purpose we have plotted in Figure 1 the relative difference $|\omega^2 - \omega_{Ch-D}^2|/\omega_{Ch-D}^2$ for the cases of p-modes for $n = 1.5, 3$ and $\ell = 2, 3$, g-modes for $n = 3$ and $\ell = 2, 3, 4$, and also for p-modes for $n = 4$ and $\ell = 2, 3$. The comparison indicates that the higher radial order the larger the relative differences are, without any dependency on ℓ value. The absolute value of relative differences found in the eigenvalues are

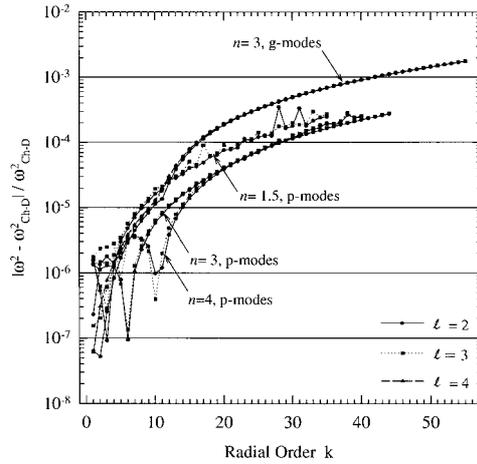


Figure 1. The absolute value of relative difference of the square of the eigenfrequency for the case of polytropes with $n = 1.5, 3$ and 4 , for $\ell = 2$ (filled circles), $\ell = 3$ (filled squares) and $\ell = 4$ (filled triangles) compared with the very accurate calculations of Christensen-Dalsgaard and Mullan (1994). The points corresponding to modes of the same degree are connected for clarity. Notice that differences between those sets of computations are larger the larger is the order k of the mode (i.e., in the case of strongly oscillating eigenfunctions).

$\lesssim 4 \times 10^{-4}$, except in the $n = 3$ g-modes, for which the difference is $\lesssim 2 \times 10^{-3}$. As Christensen-Dalsgaard and Mullan (1994) state that their computations are accurate to $\sim 10^{-8}$ we conclude that our code is able to compute the eigenvalues of polytropes with a precision of $\sim 10^{-3}$ which is enough for our purposes.

5. Application to Realistic Models: Nonradial g-Modes in Helium White Dwarfs

With the aim of investigating the behaviour of the code when it is applied to realistic models, in this section we shall consider nonradial g-modes in evolutionary models of low mass, pure helium WDs. This election is due mainly because of two reasons. First, at present we have available a detailed and updated code we have employed in the calculation of the evolution of such stars (see, e.g., Althaus and Benvenuto, 1997; Benvenuto and Althaus, 1998). Such evolutionary code is fully described in the above-cited papers. Second, the computation of nonradial modes in pure helium WDs provide us an important way for testing the pulsation code when applied to realistic models. At present, to our notice there is no study available in the literature on the pulsation properties of these objects. In evaluating our results we shall pay special attention to the asymptotic behaviour of periods of oscillation, comparing in particular the spacing of adjacent periods (for the same ℓ) with that predicted by asymptotic theory of Tassoul (1980) in stars with homogeneous composition. This theory has been employed as check for the nonradial

spectrum of polytropes (Mullan and Ulrich, 1988; Mullan, 1989) in the Cowling approximation.

5.1. OUTLINE OF PULSATION CALCULATIONS

One possible way for computing nonradial oscillations modes in evolutionary models is to first compute such structures with an evolutionary code, and then to choose a subset of these models (usually belonging to a predetermined interval in T_{eff} , called hereafter T_{eff} -strip) for pulsation analysis. This procedure has been employed in most of the studies of adiabatic pulsations in WDs to date (see, e.g., TFW, Bradley, Winget and Wood, 1993; Bradley, 1996; Brassard et al., 1992a, 1992b).

Here, we present an alternative way for computing nonradial modes in evolutionary sequences. The basic idea is very direct: if the time step in the evolutionary sequence is short enough, eigenvalues and eigenfunctions corresponding to a given model should strongly resemble those corresponding to the previous one. Then, if we couple the pulsation code to the evolutionary code, it is feasible to avoid the scan in ω^2 for each model: the search of modes is carried out only for the first model inside the relevant T_{eff} -strip. The modes are computed and stored and serve as a trial solution for the next model and so on. In this way it is possible to economize CPU time, and (more importantly) it is possible to follow the changes in the pulsational spectrum due to the evolution of the stellar structure in a continuous fashion.

5.2. DETAILS OF CALCULATIONS

Let us briefly describe how our pulsation and evolutionary codes work together. Firstly, T_{eff} -strip is chosen, as well as the frequency window to be scanned. The evolutionary code computes the model up to the moment when the model reaches the hot edge of the T_{eff} -strip. Then, the program calls the pulsation routine beginning the scan for eigenfrequencies, as described in Section 3. When a mode is found, the code generates an approximate solution for $y_i; i = 1, \dots, 4$ and ω^2 , which is improved iteratively. Then, such solution is tested and, if necessary, meshpoints are redistributed as outlined in §3, and iterated to convergence. Before improving the solution, the eigenmodes have been stored on the original grid in a common block, with the aim of being employed later as an approximate solution for the next stellar model of the sequence. When the computation of the mode has been finished, the code begins to look for eigenmodes again, and the process is repeated until the given frequency window is fully covered. Thus, we have finished the computation of all eigenmodes of the first model belonging to the T_{eff} -strip and the structure of each one is now in the computer memory. Then, the evolutionary code generates the next stellar model and the code calls pulsation routine again, but in this occasion the search for modes is skipped. Instead of this, stored eigenmodes are taken as input to the general Newton-Raphson scheme to approximate the modes of this subsequent stellar model. Then, the solution is iterated, and so on.

The whole procedure is automatically repeated for all evolutionary models inside the T_{eff} -strip. When the star gets outside the T_{eff} -strip, computation is finished.

In terms of CPU, a large fraction of the running time is spent in the search for modes, but this is executed only once. The process of relaxation and improvement of the solution is faster, and the building of each stellar model is almost instantaneous.

5.3. HELIUM WHITE DWARFS MODELS

As mentioned before, we have choose models of helium WDs for checking the efficiency our pulsation code in realistic stellar configurations. To be specific, we computed nonradial g-modes for a helium WD model with mass $M_* = 0.3M_\odot$, in the range of $8000\text{K} \leq T_{eff} \leq 25000\text{K}$. The complete sequence comprise 216 models. Convection, present in the outermost layers, is treated with the ML3 version of Mixing Length Theory.

5.4. RESULTS

Since one of our interests here is to evaluate the asymptotic behaviour of eigenperiods, we have computed nonradial g-modes for $\ell = 1, 2$ and 3 with radial order from $k = 1$ to 56 . The resulting set of modes is adequate for our purpose, since it covers a broad range in periods, enough for enabling the comparison with the predictions of the asymptotic theory of Tassoul (1980). Here we employ the mechanical external boundary condition given in Unno et al. (1989)

$$y_1 \left[1 + \left(\frac{\ell(\ell+1)}{\omega^2} - 4 - \omega^2 \right) \frac{1}{V} \right] - y_2 + y_3 \left[1 + \left(\frac{\ell(\ell+1)}{\omega^2} - \ell - 1 \right) \frac{1}{V} \right] = 0, \quad (30)$$

which replaces the first of Equations (18).

For each computed mode, the quantities of interest are the period P_k , eigenfunctions y_i ; $i = 1, \dots, 4$; kinetic energy K.E., and first order rotation splitting coefficient, $C_{\ell,k}$. In addition, we compute the variational period, P_k^v , and the weight function, WF, in the form given by Kawaler, Hansen and Winget (1985). Also, for each non-perturbed model we obtain the asymptotic spacing of periods (for the same ℓ), ΔP_A , given by (TFW, Tassoul, 1980)

$$\Delta P_A = \frac{P_0}{\sqrt{\ell(\ell+1)}}, \quad (31)$$

where P_0 is defined as

$$P_0 = 2\pi^2 \left[\int_0^1 \frac{N}{x} dx \right]^{-1}. \quad (32)$$

As in TFW, for computing P_0 we have ignored the presence of an OCZ, integrating from the center to the surface of model but setting $N = 0$ in the OCZ where

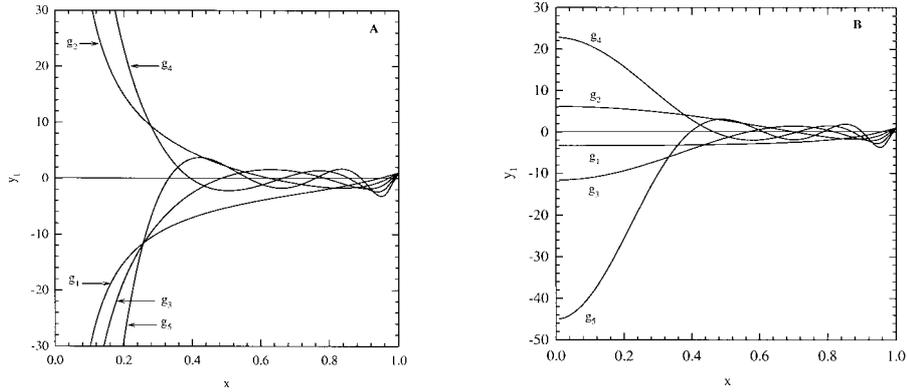


Figure 2. **A** The eigenfunction y_1 for the modes g_1, \dots, g_5 with $\ell = 1$, corresponding to the $0.3 M_\odot$ pure helium WD model with $T_{eff} = 11900\text{K}$. Notice the large amplitude of such modes in the stellar core. **B** Same as **A** but with $\ell = 2$.

$N^2 < 0$. In this approximation, we have overestimated the value of ΔP_A , but since the thickness of the OCZ (in the radial coordinate) is very small, this is a good approximation.

The results obtained for $\ell = 1, 2, 3$ are qualitatively very similar. Although pulsations have not yet been observed in these stars, in other pulsating WDs the modes with $\ell = 1$ dominate the observations, thus we shall concentrate in $\ell = 1$.

In Figure 2 we have plotted the eigenfunction y_1 for the modes g_1, \dots, g_5 with $\ell = 1, 2$, for a model with $T_{eff} = 11900\text{K}$. From Figure 2 it is evident that y_1 has large amplitudes in the whole star, especially in the core. This feature is strongly emphasized when we inspect modes with increasing k . This is a remarkable difference in the behaviour of y_1 comparing with the case of oscillating DA and DB WDs. In such WDs y_1 has a lower central amplitude (for modes with $\ell = 2$ in a $0.6 M_\odot$ DA WD model see, e.g., Figure 14a and 16a of Brassard et al., 1992b).

Figure 3 displays the normalized weight function (WF) for g_1, \dots, g_5 ($\ell = 1$) of the same model. WF provides a measure of the relative contribution from different zones in the model to the period formation. The plot indicates that period formation happens mainly in external region of star, where WF is large. However there are important contributions from central locations ($x \gtrsim 0.2$), which is somewhat different to the WF corresponding to the case of DA and DB WDs, for which WF displays appreciable values only in the envelope (in the ‘normal’ modes; see Figures 15 and 17 of Brassard et al., 1992b).

Let us discuss our results concerning periods (P_k) and kinetic energies (K.E.). The periods here computed show the expected trend in g-modes, with values increasing with the order of modes k . Since our calculations covers a wide T_{eff} -strip, it is possible infer the changes of periods during WD cooling.

It is worth mentioning that we have computed the Brunt-Väisälä frequency as in TFW (the Ledoux term B is zero for the whole of model, since it is chemic-

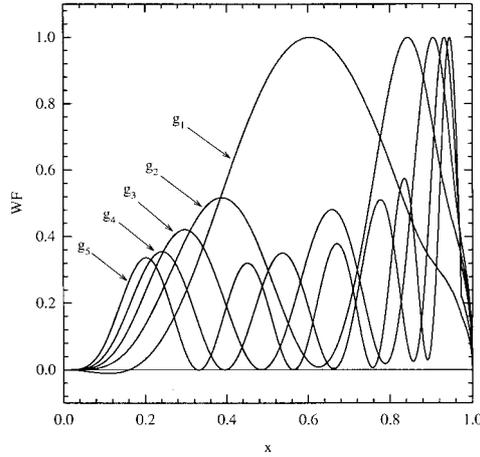


Figure 3. The normalized weight function WF corresponding to the same modes included in Figure 2A.

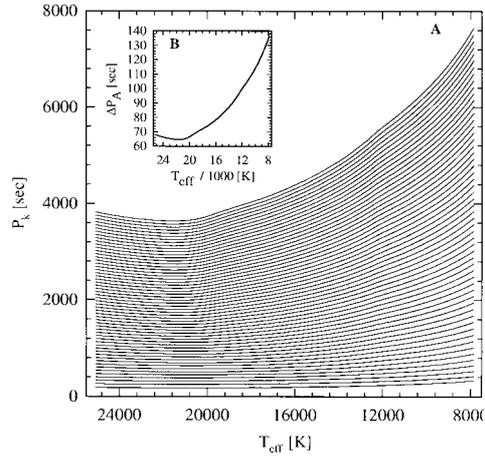


Figure 4. **A** Periods of dipolar modes ($\ell = 1$) with radial order from $k = 1$ to $k = 56$ and **B** the asymptotic period spacing ΔP_A predicted by Tassoul (1980) theory as a function of the effective temperature. Notice that the periods of high order modes have a very similar behaviour when compared with ΔP_A during WD cooling.

ally homogeneous). We have found that N^2 decreases as the model cools down, a feature that is very pronounced in the core. It is due mainly to the increasing degeneracy in that region, decoupling more and more pressure from temperature (χ_T drops). As consequence eigenperiods increase monotonically when T_{eff} drops below ~ 20000 K, as clearly shown in Figure 4A.

Next, we consider the behaviour of the kinetic energy (K.E.) of the modes. We have found that, as expected in a chemically homogeneous star, K.E. is a smooth

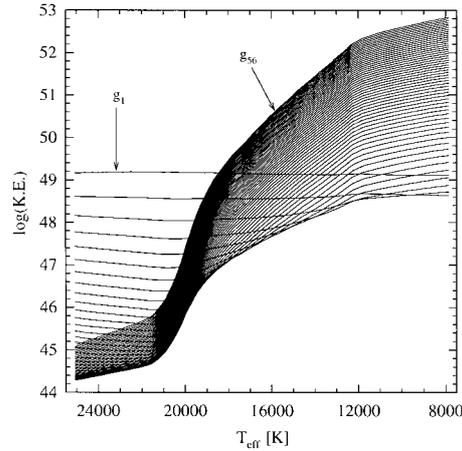


Figure 5. Kinetic energy (K.E.) vs. T_{eff} for the same modes with $\ell = 1$ included in Figure 4. The unit of K.E. is ergs and we have assumed that the amplitude of y_1 at surface is one. For more details, see text.

function of k and thus of the period. In Figure 5 we show $\log(\text{K.E.})$ vs. T_{eff} for the models included in Figure 4. For hotter models it is clear that the more energetic modes are those with lower order, because they penetrate deep in the star, where density is high. However, when T_{eff} drops below ~ 20000 K the energy of high order modes (which are mainly concentrated in the outer envelope of star) strongly increase. The explanation of this effect (see Brassard et al., 1992b in context of a DA WD star) resides in the fact that at such T_{eff} value the OCZ of the object suddenly gets thicker (see the dotted line in Figure 17 of Althaus and Benvenuto, 1997). The high order modes that oscillates in the envelope feel gradually the presence of convection as the star cools down. Since that in OCZs g-modes become evanescent, such modes are forced to get larger amplitudes somewhat below the bottom of the OCZ where the density is larger. Since K.E. is proportional to the integral of square of displacement, weighted by ρ , these modes oscillate with larger energies. In sharp contrast to the behaviour of high k modes, low order modes are rather insensitive to the thickening of the OCZ.

Now let us discuss the period spacing of the modes. To our knowledge there is no study available in the literature devoted to nonradial pulsations of helium WDs. This situation does not allow us to perform a direct comparison of our results. Nevertheless we can have a good idea related to the reliability of our pulsation code examining the period spacing of consecutive modes of the same ℓ . Tassoul (1980) predicted that the asymptotic period spacing for g-modes for a chemically homogeneous object in the adiabatic approximations is a constant value ΔP_A (see Equation 31). In Figure 6 we show the forward period spacing $\Delta P_k = P_{k+1} - P_k$ vs. radial order k ($\ell = 1$) for models with $T_{eff} = 13200, 11800$ and 9400 K. As reference we also include in this figure the corresponding asymptotic spacing

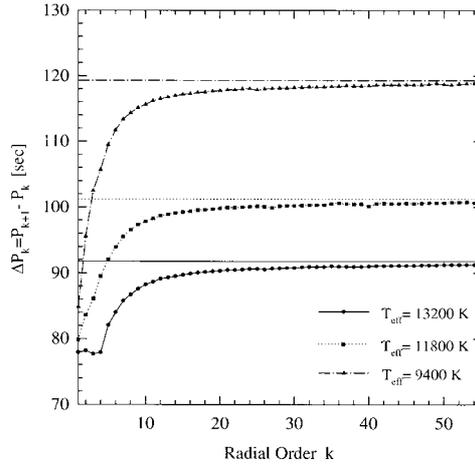


Figure 6. Forward period spacing (ΔP_k) vs. radial order, for modes with $\ell = 1$ for three models with different T_{eff} values. Symbols corresponding to modes of the same T_{eff} are connected for clarity. Horizontal lines correspond to the value of asymptotic period spacing in each effective temperature according to Tassoul (1980).

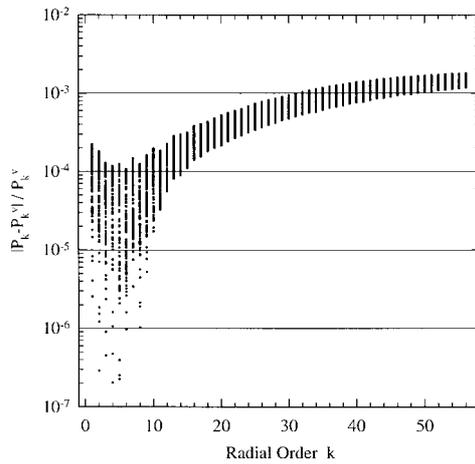


Figure 7. The absolute value of relative difference between numerically computed and variational eigenperiods for modes with radial order from $k = 1$ to $k = 56$ and $\ell = 1$ in all models analyzed.

predicted by Tassoul (1980) treatment. From Figure 6 it is clear that the trend of the numerical solution is the correct one.

In order to get an estimation of the accuracy of our treatment, we have calculated the variational eigenperiods (P_k^v) for each mode in the formalism presented by Kawaler et al. (1985). The comparison of variational and numerical eigenperiods for the values of ℓ here considered gives a difference between these treatments which is lower than $\approx 1\%$ (see Figure 7 for the case of $\ell = 1$).

By the way, let us comment on that we have also tested our code running it on the same models for which oscillation modes were previously computed. Specifically we have employed for such test two carbon-oxygen DA WD models of $0.5 M_{\odot}$ and $0.85 M_{\odot}$ whose structure was computed with the WDEC evolutionary code and its vibrational properties were previously analyzed (Bradley, 1996). In this case $M_H/M_* = 10^{-4}$; $M_{He}/M_* = 10^{-2}$ and $T_{eff} \approx 12500$ K. This is a valuable test for our pulsation code because we could confront our results with those of other researchers working on detailed models. Also the fact that these models were computed with the same evolutionary code is valuable because differences in the resulting periods are due only to the pulsation code and not to differences in e.g. between the treatment of the equation of state in our evolutionary code and in WDEC. Considering a large amount of modes, the differences between the previously computed periods and those we have found has been below $\approx 0.1\%$. Thus, we judge that our code produce results accurate enough for our purposes. This comparison has been made possible by Paul Bradley who provided us with the DA WD models and the periods of many its modes.

6. Summary and Conclusions

In this paper we have presented a general code intended for calculating linear, adiabatic, nonradial eigenfunctions and eigenfrequencies corresponding to spherically symmetric stars by means of a finite difference scheme. This is a very simple strategy based on a slight modification of the general Newton-Raphson method for computing stellar evolution presented by Kippenhahn, Weigert and Hofmeister (1967) because of the different structure of the matrix to be inverted in finding the iterative corrections.

As a first test to this code we have applied it to the simplest case: polytropic spheres. As there exist very precise tabulations of the eigenfrequencies for such configurations, we have been able to perform a quantitative comparison of our results with the best available in the literature (Christensen-Dalsgaard and Mullan, 1994). We have found that our code is able to compute polytropic eigenfrequencies with an accuracy of 10^{-3} or much better for low k values.

Then, we have coupled this nonradial oscillation code to our evolutionary code and applied it to the computation of the pulsation spectrum of pure, low mass helium white dwarfs (WDs). Specifically we considered the case of an object with a mass of $0.3 M_{\odot}$ for a wide range of effective temperatures (T_{eff} -strip). In making such computation we have employed a new method. Taking a short-enough time step in the evolutionary sequence, eigenvalues and eigenfunctions corresponding to a given model strongly resemble those corresponding to the previous one. Then, we scan for modes in the first model inside the T_{eff} -strip. Eigenmodes are computed and stored and serve as a trial solution for the next model and so on. In this way it is

possible to follow the changes in the the pulsational spectrum due to the evolution of the stellar structure in a continuous and fairly detailed fashion (see Figure 4).

Regarding the computation of g-modes for the $0.3 M_{\odot}$ pure helium WD model we have presented eigenfunctions and weight function (WF) for low order k modes, and periods, period spacing, kinetic energy (K.E.) and variational periods for a wide range of modes. These quantities were computed for the case of $\ell = 1, 2$ and 3 but in Figures 3–7 we have presented only the case $\ell = 1$ for brevity. To our notice this is the first effort in studying the pulsation properties of helium WDs.

Our experience with realistic helium WD models indicates that our numerical code for pulsations work nicely and in particular the solutions we have found have an asymptotic period spacing in good agreement with the predictions of the analytical theory of Tassoul (1980). This, and the fact that comparison with more complex WD models with layers of different internal composition show us that our numerical tool for computing eigenmodes work adequately, especially regarding the coupling we have performed between it and our evolutionary code. We plan in the near future to apply this new code to the study of pulsation of WDs in general.

Acknowledgements

The authors would like to warmly acknowledge Paul Bradley for his kindness in providing us with numerical models of carbon-oxygen DA WDs together with his pulsation calculations and also for the time he spent in doing so. This allowed us to perform key a test to our code, such that it made possible to us to be confident with the results the code produces. We also want to thank our referee D. Winget for suggestions that allowed us to improve the original version of this paper.

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